# Weak Dominance

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## 1 Weak Dominance

**Example 1.1** (Referendum)**.** Consider a game where there are 100 villagers as players. The set of strategies to vote or not. The utilities are given as:

$$\begin{cases} 1 & \text{succeeds if 50+ votes} \\ 0 & \text{otherwise} \end{cases}$$

Assume that all villagers would like to change the policy and there is no cost to voting. It is clear the right way to play this game is to vote. Taking a look at the possible outcomes:

- If 50+ vote, then your actions don't matter
- If 50 others vote, then voting gives 1, and not voting gives 0
- If <50 vote, then your actions don't matter

It is clear that you can never do better than voting, but in some cases, it does not matter. Note that voting is **not strictly dominant** by definition. □

**Definition 1.1** (Weak Dominance)**.** *We say that $s_i$ weakly dominates $s_i'$ if:*

$$U_i(s_i, s_j) \geq U_i(s_i', s_j) \;\; \forall \;\; s_j$$

$$U_i(s_i, s_j) > U_i(s_i'.s_j) \text{ for some } s_j$$

*$s_i$ is weakly dominant if it weakly dominates ever other $s_i'$.*

An interpretation of a weakly dominant strategy is that it is a *safe bet*, where you can't do any worse by playing it.

**Theorem 1.1** (Weak Dominance and Best Response)**.** *If $s_i$ is weakly dominant, then $s_i \in BR_i(\sigma_j)$ for the subset of beliefs $\sigma_j$. If $\sigma_j$ assign strictly positive probability to every $s_j$, then $BR_i(\sigma_j) = \{s_i\}$. In essense, if you assign a positive probability to every possible strategy, then the weakly dominant strategy is the unique best response.*

**Proof** Using the definition of best response, we have that:

$$U_i(s_i, \sigma_j) = \sum_{s_j} u_i(s_i, s_j) \cdot \sigma_j(s_j) \geq \sum_{s_j} u_i(s_i', s_j) \cdot \sigma_j(s_j) \because u_i(s_i, s_j) \geq u_i(s_i', s_j) \;\; \forall s_i'$$

As well, at least one $u_i(s_i, s_j) > u_i(s_i', s_j)$ and the weights are greater than 0. ■

## 2 Joint Policy

Consider a game with 3 bankers in the Federal Reserve to set an interest rate, $x$, s.t. $x \in \{0..10\}$. Each banker, denoted as $i \in \{1, 2, 3\}$ has an favorite interest rate $x_i$, s.t. $x_1 = 2 \wedge x_2 = 4 \wedge x_3 = 7$. Every banker $i$ proposes policy $s_i \in \{0..10\}$ and the median policy $s$ is chosen, such that if $s_1 < s_2 < s_3$, then $s = s_2$. The payoffs of the game are given as $-|x_i - s|$. **In this game, we note that the weakly dominant strategy is to choose each one's favorite strategy.**

**Proof** Define $\underline{s}$ as $\min(s_{i-1}, s_{i+1})$ and $\bar{s}$ as $\max(s_{i-1}, s_{i+1})$. Taking the game, we can derive the a player's utility cases as:

$$u_i(s_i, \underline{s}, \bar{s}) = \begin{cases} -|x_i - \underline{s}| & s_i \leq \underline{s} \\ -|x_i - s_i| & s_i \in \{\underline{s}..\bar{s}\} \\ -|x_i - \bar{s}| & s_i \geq \bar{s} \end{cases}$$

Considering the three possible cases:

- If $x_i < \underline{s}$, then $\max(u_i) = -|x_i - \underline{s}|$ if you report $s_i = x_i$.
- If $x_i \in \{\underline{s}, \bar{s}\}$, then $\max(u_i) = 0$ and reporting $s_i = x_i$ is the unique best response.
- If $x_i > \bar{s}$, then $\max(u_i) = -|x_i - \bar{s}|$ and if you report $s_i = x_i$

Therefore, $\forall\ \{\underline{s}, \bar{s}\}, s_i = x_i \implies max(u_i)$ and therefore is the unique best response and is sometimes strictly best. Therefore, $s_i = x_i$ **is weakly dominant**. ■

However, taking a modified version of the game, suppose that instead of the median being implemented, *the average of the three is reported.*

**Question 2.1.** Would reporting truthfully in this modified game still be weakly dominant?

**Example 2.1** (Modified Joint Policy)**.** Suppose that $x_i = 3$, $x_{i+1} = 10$, and $x_{i-1} = 10$. We see the following cases:

- If $s_i = 3$, then $s = \frac{22}{3} \implies u_i = -\left|3 - \frac{22}{3}\right| = -\frac{14}{3}$
- If $s_i = 0$, then $u_i = -\frac{11}{2}$

Therefore, $s_i = x_i$ is **not** weakly dominant. □

## 3 2nd Price Auction

Consider an increasing price auction, with each buyer having a value in mind for the item, which is internal to the buyer. Each buyer submits a bid for the item (which can by any value), and the highest bid is chosen as the winner. The winner must pay the second-highest bid. This is similar to proxy auto-bids on eBay. To formalize, we denote the players as 100 students with value $V_i$ and $S_i = \{0..\infty\}$ as bid, $b_i$. We define the utility function as:

$$u_i(b_i, b_j) = \begin{cases} V_i - max(b_j) & b_i > max(b_j) \\ 0 \end{cases}$$

In case of a tie, a winner will be chosen randomly from the two highest bids. The weakly dominant strategy in this game would be to bid $b_i = V_i$.

**Proof** Consider $b_i' > b_i = V_i$. We have multiple cases as:

- Suppose $\max(b_j) > b'_i \wedge b_i \implies u_i = 0 \ \forall \ b_i \wedge b'_i$
- Suppose $\max(b_j) \in \{b_i..b'_i\}$. You have the following utilities:

$$\begin{cases} 0 & b_i \\ V_i - max(b_j) = b_i - max(b_j) < 0 & b'_i \end{cases}$$

- Suppose $\max(b_j) < b'_i \wedge b_i \implies u_i = v_i - \max(b_j) \ \forall \ b_i \wedge b'_i$

Note that $b_i$ weakly dominates any $b'_i > b_i$. A similar example can be shown for $b_i < b'_i$. This shows that $b_i = V_i$ is **weakly dominant**. ■

**Question 3.1.** In a first price auction, is bidding truthfully still weakly dominant?

If you know the bids of all the other players, you should bid simply slightly higher than the second-highest bid. Therefore, we can only show that bidding truthfully is not weakly dominant.