Rationalizability

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April 10, 2018

1 Beauty Content

Suppose there are players which are the set of all 106G students. The space of strategies are $S_i = \{1, 2, ..100\}$. Imagine there is a jury and each player picks a number which they believe is the most beautiful. Take the utilities to be:

$$U_i(s_i, s_j) = -\left|s_i - \frac{2}{3}\bar{S}\right|$$

In this scenario, the further away your choice is from two-thirds of the average, the worse off you are. Your goal is to get to exactly two-thirds of the average. Take it s.t. if you think everybody picks $s_i = 100$, then you pick 67. If you think everybody picks randomly from the set of all possible strategies, $E[S_i] = 50$, and you should choose 33. If you think everybody choose $S_i = 1$, then you also pick 1.

It is important to note that $s_i = \{1..67\}$ are all best responses and $s_i = \{68..100\}$ are strictly dominated by $s_i = 67$. We can show this as:

$$U_i(69, s_j) = -\left|69 - \frac{2}{3}\bar{S}\right| = -|69 - 67| - \left|67 - \frac{2}{3}\bar{S}\right| < \left|67 - \frac{2}{3}\bar{S}\right| = U_i(67, s_j)$$

This logic additionally applies to all $s_i \geq 68$. If you are a rational player, then you will pick $s_i \in \{1..67\}$. However, if all players are rational and pick within the set $s_i \in \{1..67\}$, you will again lose another $\frac{2}{3}$ of the set of possible rational outcomes, resulting in $s_i \in \{1..45\}$. If iterated ad infinitum, this will cause the loss of all possibilities in the set except for $s_i = 1$. *It is important to know that there is a common knowledge of rationality among all players in the game.* In essence, this can be justified through a game ad infinitum where a "best response to last round" model is applied.

2 Rationalizability

Definition 2.1 (Rationalizability)**.** *Let $BR_i^1 \subseteq S_i$ that are best responses to some beliefs. For the previous Beauty Contest game, this includes:*

$$BR_i^1 = \{1..67\}$$

We can then let $BR_i^2 \subseteq BR_i^1$ that are best responses if others play BR_i^1, i.e. $BR_i^2 = \{1..45\}$. This can be done for every n to:

$$BR_i^{n+1} \subseteq BR_i^n \text{ that are BR's if others play } BR_j^n$$

A strategy s_i is ***rationalizable*** *if $s_i \in BR_i^n$ for every n. In the beauty content, $s_i = 1$ is rationalizable. If each player has a unique rationalizable strategy in a game, this game is known as* ***dominance solvable****. The Beauty Content game is dominance solvable.*

Example 2.1. Consider the modified prisoner's dilemma:

		B	
		C	D
A	C	(2,2)	(0,3)
	D	(3,0)	(1,1)

Taking this, we consider $BR_A^1 = \{D\}$, which is identical to the standard prisoner's dilemma. For player B, this is the same. As we are left to only one element in each set of best responses, we conclude this game is dominance solvable in one step. □

Example 2.2 (Reciprocator 1)**.** Consider another modified prisoner's dilemma where B is a reciprocator:

		B	
		C	D
A	C	(2,**4**)	(0,3)
	D	(3,0)	(1,1)

For player A, the payoffs are exactly the same and there is no impact. Therefore, $BR_A^1 = \{D\}$. For player B, $BR_B^1 = \{C, D\}$. However, after a single iteration, we have that $BR_A^2 = \{D\}$ and $BR_B^2 = \{D\}$. Player B will assume that Player A will play D, and therefore should choose D. Therefore, this game is dominance solvable after 2 iterations. □

Example 2.3 (Reciprocator 2)**.** Consider another modified prisoner's dilemma as follows:

		B		
		C	X	D
A	C	(2,2)	(1,1)	(0,1)
	D	(3,0)	(0,-1)	(1,1)

It is key to note that there is no already a dominant strategy in this game immediately for either player. However, it is clear that $s_B = X$ is strictly dominated in all cases by $s_B = C$. We can see that $BR_A^1 = \{C, D\}$. If $s_B = C, X$, you will rather play $s_A = C$. You will play $s_A = D$ if $s_B = D$. For player B, we have $BR_B^1 = \{C, D\}$. If A defects, $s_B = D$. If A cooperates, $s_B = C$.

Note that X does **not** factor into this game anymore. It can never be a best response since it is strictly dominated by C. If a second iteration is performed, this follows the same pattern as **Example 2.2** with the modified dilemma with reciprocator.

$$BR_A^2 = \{D\} \wedge BR_B^2 = \{C, D\} \implies BR_A^3 = \{D\} \wedge BR_B^3 = \{D\}$$

□

3 Electoral Competition

Imagine two players, D and R, with strategy sets $s_i = \{1..9\}$ where the numbers represent the amount of conservatism in the strategy. We will define the utility function to be that each politician cares about *only* number of votes. Suppose:

$$\forall\ s_i\ \exists\ 100 \text{ votes}$$

Assume that each voter votes for the party whose platform is closest to his/her ideal point. As an example, we see that $U_D(3,6) = 400$ since voters with beliefs $s_i = \{1..4\}$ would choose D, and therefore $U_R(6,3) = 500$. However, if voters are indifferent between D and R, they split 50-50. For example, $U_D(3,5) = 350$.

Example 3.1 (Dominance Solution)**.** Moving iteratively, we can see that $s_i = 1$ is strictly dominated by $s_i = 2$. We can see that

$$U_D(1, s_i)\ \forall\ i \in \{1..9\} < U_D(2, s_i)$$

This is the mirrored for $s_i = 9$ which is strictly dominated by $s_i = 8$. However, if taken into iteration, we can eliminate the extreme values again, and so on. Therefore, ad infinitum, all extreme values are removed and the game is **dominance solvable** with $s_i = 5$ in four steps. □