Pivot Mechanisms

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1 Second Price Auction

Remember that the second price auction showed that bidding truthfully is a weakly dominant strategy and that if everyone plays the weakly dominant strategy, then the allocation i9s *efficient*. In this case, efficiency is defined as *the person who values the good most gets the good*. This is an example in a broader class of games with pivot mechanisms.

2 Abstract Allocation Problem

Consider the following allocation problem, with the set of allocations $\{x_1..x_m\}$ and a group of players, denoted I. Each player $i \in I$ has the utility function $u_i(x_m, t_i) = v_i(x_m) - t_i$ when the allocation x_m is chosen and is asked to pay t_i (which is independent of price). Otherwise, this noted as:

$$u_i(x_m, t_i) = \begin{cases} v_i - t_i & x_m = x_i \\ 0 - t_i & x_m \neq x_i \end{cases}$$

For clarification, consider a problem of choosing which players get an object. There are I allocations (see combinatorics) for who gets the object. If player $i \in I$ gets the object, this causes utility $v_i(x_m)$. Otherwise, he gets zero. Additionally, to get the object, he must pay t_i, which subtracts from his utility.

Definition 2.1 (Efficiency). *To be efficient, you want to pick an* ***efficient allocation*** x_i *that maximizes the sum of the utilities without taking into account price. This is formalized as:*

$$\sum_{i \in I} v_i(x_m) \geq \sum_{i \in I} v_i(x'_m) \ \forall \ x'_m \neq x_m$$

It's important to note that money is not a part of this definition of efficiency. Recalling from ECON 101, money does not play a role in efficiency in general as it is a redistribution of resources. Take this definition of efficiency as a given. Note that the sum is the aggregate utility of all players in the game.

Looking at the second price auction, $sum_{i \in I} v_i(x_m) = v_m$ as all other utilities in the game are simply 0. Therefore, for the second price auction, our objective is simply $v_m \geq v'_m$. It is efficient to get the good to the person who values it most.

3 Pivot Mechanisms

Definition 3.1 (Pivot Mechanisms)**.** *A pivot mechanism is s.t.:*

- *Each player i reports a utilities s.t.:*

$$\{\tilde{v}_i(x_i)..\tilde{v}_i(x_m)\}$$

  where $\tilde{v}_i$ denotes the value reported

- *$\exists$ social planner that chooses allocation s.t.:*

$$\sum_{i \in I} \tilde{v}_i(x_m) \geq \sum_{i \in I} \tilde{v}_i(x'_m) \ \forall \ x'_m \neq x_m$$

  taking the reports at face value and picking the efficient allocation following those

- *Determine payments to be s.t.:5*
  1. *For each player i, determine his* ***pivotal type****:*

$$\{\tilde{v}_i^P(x_1)..\tilde{v}_i^P(x_m)\}$$

     that would just change the allocation

  2. *Player i players $t_i = \tilde{v}_i^P(x_m)$ where x_m is the allocation chosen in* ***(2)***

  In the second price auction, if i wins the good with $\tilde{v}_i$ being highest, the pivotal type would be the report of i that would just result in a change in allocation (e.g. someone else gets the good), which is i.e. the second highest bid.

Given this definition, we have a resulting theorem regarding the relationship between weak dominance and pivot mechanisms:

Theorem 3.1 (Weak Dominance and Pivot Mechanisms)**.** *In a Pivot mechanism-type game, it is* ***weakly dominant*** *for all players to report $\{\tilde{v}_i(x_1)..\tilde{v}_i(x_m)\} = \{v_i(x_1)..v_i(x_m)\}$ which will result in an efficient allocation that satisfies* ***Definition 3.1****.*

3.1 Public Good

Suppose that you are an R.A. in the dorm s.t. you must decide whether to purchase a PlayStation or not (defined as b or n). This results in allocation set $\{b, n\}$. You pay \$1000 and that you gain no utility from having it or you pay \$0 otherwise. As well, $\exists i \in \{1..100\} : v_i = \{1..31\}$. Formalized:

$$\begin{aligned}
u_0(x_b, t_0) &= -1000 - t_0 \\
u_0(x_n, t_0) &= 0 - t_0 \\
u_i(x_b, t_i) &= v_i - t_i \\
u_i(x_b, t_i) &= 0 - t_i
\end{aligned}$$

We begin to define the pivot mechanism as follows;

1. All $i \in \{1..100\}$ reports utility from buying PlayStation

2. Assuming reports are true, our efficient allocation is

$$\begin{cases} x_b & \sum_{i\in\{1..100\}} \tilde{v_i}(x_b) \geq 1000 \\ x_n & \sum_{i\in\{1..100\}} \tilde{v_i}(x_b) < 1000 \end{cases}$$

3. If we do not define strict payments, everyone is incentivized to report an untruthfully high value. Therefore, we define payments as:
   (a) If n, you pay 0. This is a given.

   (b) If $\sum_{i\in\{1..100\}} \geq 1000$, what is the pivotal type of player 3? Suppose that $\sum_{i\in\{1..100\wedge\neq 3\}} = 997$ and $\tilde{v_3} = 7$, then $\tilde{v}_3^P = 3 \implies t_3 = 3$. Suppose that $\sum_{i\in\{1..100\wedge\neq 3\}} \geq 1000$ then $t_3 = 0$ since there is nothing that could be done to change the allocation.

Looking at **Theorem 3.1**, suppose that $v_3(x_b) = 7$. We have the following utilities:

		$\tilde{v_3}$			
		5	**6**	**7**	**8**
	991	0	0	0	0
	992	0	0	0	$7-8=-1$
$\sum_{i\neq 3}\tilde{v_i}$	**993**	0	0	$7-7=0$	$7-7=0$
	994	0	$7-6=1$	$7-6=1$	$7-6=1$
	995	$7-5=2$	$7-5=2$	$7-5=2$	$7-5=2$

Note that reporting 7 is weakly dominant in this game table. As well, note that payments might be differ between the set of all players. Suppose that $\tilde{v_1} = 10, \tilde{v_2} = 5, \sum_{i>3}\tilde{v_i} = 992$. Determining payments for $i = 1$ is as follows:

$$\sum_{i\neq 1} \tilde{v_i} = 997 \implies t_i = 3$$

For player 2:

$$\sum_{i\neq 2} \tilde{v_i} = 1002 \implies t_i = 0$$

3.2 k goods for sale

Consider a seller with 6 units (k) of private goods to allocate. These are strictly private goods. Suppose there $i \in \{1..100\} : \forall i \; \exists \; v_i$. As well, suppose that each player only gets 1 unit, and therefore, the possible allocations are $100 \cdot 99 \cdot ...(100-k)$.

It is obvious that the efficient allocation is that the goods go to the k highest values. We determine the payments as:

1. If you don't get a good, $t_i = 0$

2. If you get a good, your pivotal type is the $k+1$ highest value

Note that this is essentially a $k+1$ price auction.

3.3 Trade

Suppose there are two players, owner with value v_s and potential buyer, v_b. There are two possible allocations: $\{x_{trade}, x_{notrade}\}$. Note that the owner wants to inflate v_s and the buyer wants to reduce v_b, to improve their own situations. Using a pivot mechanism, we determine the efficient allocation to be $\begin{cases} x_t & v_b > v_s \\ x_{nt} & v_s > v_b \end{cases}$. For the payments, we have:

1. If no trade, then $t_i = 0$
2. If we trade, we determine the pivotal type for the buyer, $\tilde{v}_b^p = \tilde{v}_s$. As long as $v_b > v_s$, the trade will happen, but anything lower, it will not. The mirror is true for the seller. **Note that if $\tilde{v}_b^p = 10$ and $\tilde{v}_s = 5$, then the buyer pays 5 and the seller gets 10**. Note that this is not balanced, which suggests that there must be an outside party to balance the transaction.

It's important to see that the transfers merely serve the purpose of incentivizing the players to report truthfully, not to balance a budget.