Mixed Strategies and Nash Equilibrium

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1 Examples

1.1 Matching Pennies

		2: a	b
1	a	-1,1	1,-1
	b	1,-1	-1,1

Note that when attempting to find a Nash equilibrium in pure strategies, none exist.

1.2 Rock, Paper, Scissors

		R	2: P	S
	R	0,0	-1,1	1,-1
1	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

Note, it is clear here that there is also no pure strategy NE in this game either.

Definition 1.1 (Mixed Strategy)**.** *A mixed strategy σ_i is a probability distribution over S_i. $P(s_i \in \sigma_i) \coloneqq \sigma_i(s_i)$. We define the expanded strategy set to be ΔS_i that consists of all possible randomization over S_i.*

In the **Matching Pennies** example, we define the probabilities to be $p \mid p \in [0,1]$. In **RPS**, we define the probabilities as $(p, q, 1-p-q) \mid p \geq 0 \ \wedge \ q \geq 0 \ \wedge \ p+q \leq 1$.

Remark 1.1. Note that in a standard pure strategy is a proper subset of a mixed Nash Equilibrium. Considering **MP**, $a = \sigma_1 : \sigma_1(a) = 1$. △

Definition 1.2 (Mixed Strategy BR)**.** *A mixed strategy σ_i is a best response to σ_j if $\forall \sigma_i(s_i) > 0, s_i \in BR(\sigma_j) \implies$:*

$$U_i(s_i, \sigma_j) \geq U_i(s_i', \sigma_j) \ \forall s_i'$$

It is important to note the *redefinition* of σ_j. We are now denoting σ_j to be the distribution of other player's strategies. This is mathematically identical to the previous notation of σ_j representing the beliefs of others.

Definition 1.3 (Mixed Nash Equilibrium)**.** *A profile of mixed strategies* $(\sigma_1^*..\sigma_N^*$ *is a NE if* $\forall \sigma_i^* \in BR(\sigma_j^*)$.

Question 1.1. Can a strictly dominated strategy be played with positive probability in a Nash Equilibrium of mixed strategies?

No. A strictly dominated strategy can never be part of the best response. Therefore, it can not be included in the definition of a Mixed Strategy BR (see Def. 1.2).

Theorem 1.1. *If* $\sigma_i(s_i) > 0$ *in a mixed strategy NE, then it is a rationalizable.*

Theorem 1.2. *Every game has a mixed strategy Nash Equilibrium.*

2 Finding Mixed Strategy Nash Equilibriums

2.1 Matching Pennies

Suppose that Player 1 uses a mixed strategy:

$$\sigma_i = \begin{cases} a & P(a) = p \\ b & P(b) = 1 - p \end{cases}$$

We proceed by discovering $BR_2(\sigma_1)$:

$$\begin{cases} U_2(a, \sigma_1) = 1p + -1(1-p) & = 2p - 1 \\ U_2(b, \sigma_1) = -1p + (1-p) & = 1 - 2p \end{cases}$$

$$BR_2(\sigma_1) = \begin{cases} a & 2p - 1 > 1 - 2p \implies p > \frac{1}{2} \\ b & p < \frac{1}{2} \\ \{a, b\} & p = \frac{1}{2} \end{cases}$$

We can then pivot the discussion to Player 1, which is essentially quasi-symmetric in this scenario:

$$BR_1(\sigma_2) = \begin{cases} a & 2q - 1 < 1 - 2q \implies q < \frac{1}{2} \\ b & q > \frac{1}{2} \\ \{a, b\} & q = \frac{1}{2} \end{cases}$$

We then define the mixed strategy Nash Equilibrium (σ_1^*, σ_2^*) to be $\sigma_1^*(a) = \frac{1}{2} \wedge \sigma_2^*(a) = \frac{1}{2}$. **It's important to note that in a mixed strategy Nash Equilibrium, Player i mixes to keep his opponent indifferent**.

2.2 Market Entry Game

Consider players to be 4 firms, A..D with $S_i = \{\text{Entry}, \text{No Entry}\}$. The payoffs are defined as:

- $u_i(N, s_{-i}) = 0$
- $u_i(E, s_{-i}) = \pi(s) - c_i$

We can further define c_i as

$$\begin{cases} 100 & c_A \\ 300 & c_B, c_C \\ 500 & c_D \end{cases}$$

As well, we define $\pi(s)$ as a function of number of firms entered

$$\begin{cases} 1000 & \pi(1) \\ 400 & \pi(2) \\ 250 & \pi(3) \\ 150 & \pi(4) \end{cases}$$

We begin by taken the lowest hanging fruit. Nobody entering the market is strictly dominated by entering the market. The same is true for everyone entering. Therefore, the extreme situations are not Nash Equilibriums. As well, it is clear that Firm A should *always* enter and is strictly dominant. Given its low costs c_A, it will always make a profit in $\pi(n) \mid n \in \{1..4\}$. Therefore, Firm A enters in **every** Nash Equilibrium.

Next, taken firm A's enter as a given, Firm D should never enter. Taking $\pi(n) \mid n > 1$, Firm D will only lose if it chooses to enter. Therefore, with the choices for A and D done, the game has been effectively reduced to a game with players B and C. Therefore, we are left with:

		C	
		E	N
B	E	-50,-50	100,0
	N	0,100	0,0

Note that there exist 2 pure strategy Nash Equilibriums (Chicken Game - anti-cooperation), in the disjoint strategy sets, (E,N) and (N,E). As well, we can attempt to solve of mixed strategy NE's.

Let $\sigma_c \coloneqq \sigma_c(E) = q$. Therefore, Firm B would take on:

$$\begin{cases} U_B(N, \sigma_C) & = 0 \\ U_B(E, \sigma_C) = -50q + 100(1-q) & = 100 - 150q \end{cases}$$

Therefore:

$$BR_B(\sigma_C) \begin{cases} \{N, E\} & q = \frac{2}{3} \\ E & q < \frac{2}{3} \\ N & q > \frac{2}{3} \end{cases}$$

Note that this game is symmetric (not quasi-symmetric) for both players. Therefore, there is a mixed strategy Nash Equilibrium $(E, \sigma_B^*, \sigma_c^*, N) \;\mid \sigma_B^*(E) = \frac{2}{3} \;\wedge\; \sigma_C^*(E) = \frac{2}{3}$.

2.3 Modified Matching Pennies

		2: a	2: b
1	a	-3,1	1,-1
	b	1,-1	-1,1

Again, it is clear that this game has no pure strategy Nash equilibrium. We can define the mixed Ne from considering Player 2:

$$BR_2(\sigma_j) = \begin{cases} \{a,b\} & \sigma_1(a) = \frac{1}{2} \\ \ldots \end{cases}$$

Suppose Player 2 players $\sigma_2 \coloneqq \sigma_2(a) = q$. Therefore:

$$\begin{cases} U_1(a,\sigma_2) = -5q + 1(1-q) & = 1-6q \\ U_1(b,\sigma_2) = 1q + (-1)(1-q) & = 2q-1 \end{cases} \implies BR_1(\sigma_2) = \begin{cases} \{a,b\} & = 1-6q = 2q-1 \to q = \frac{1}{4} \\ \ldots \end{cases}$$

Therefore, the Nash Equilibrium in mixed strategies is $(\sigma_1^*, \sigma_2^*) \mid \sigma_1^*(a) = \frac{1}{2} \wedge \sigma_2^*(a) = \frac{1}{4}$.