Dominance and Best Response

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1 Game Theory Basics

Definition 1.1. *A normal form game consists of:*

- *Players* $i \in 1..n$
- *For every player* i*,* $\exists$ *set of strategies* S_i *such that* $s_i \in S_i$
- *For every player* i *and every profile of strategies* $S = \{s_1, ..., s_n\}$ *or* $S = \{s_i, S_j\}$ *such that* $s_j = \{s_1, ...s_{i-1}, s_{i+1}, ...s_n\}$

Notation 1.1. *If there are two players,* i *and* j*, then* $U_i(s_i, s_j)$ *and* $U_j(s_j, s_i)$*. Note that discussions of utility and strategy choices always take the player in perspective as the first argument and the other players are subsequent arguments.*

2 Prisoner's Dilemma

There exists two players, 1 and 2. Their game theory payoffs can be defined as follows:

		2: C	2: D
1	C	-1, -1	-5, 0
	D	0, -5	-3, -3

Suppose that you are Player 1. You will make the following call:

If $S_2 = C$, then you will have $\begin{cases} U_1(C, C) = -1 \\ U_2(D, C) = 0 \end{cases} \implies S_1 = D$

If $S_2 = D$, then you will have $\begin{cases} U_1(C, D) = -5 \\ U_2(D, D) = -3 \end{cases} \implies S_1 = D$

If you suppose that 1 thinks that $P(S_2 = C) = q$ and $P(S_2 = D) = 1 - q$. We have that

$$\begin{cases} U_1(C, q) = q(-1) + (1-q)(-5) & = 4q - 5 \\ U_2(D, q) = q(0) + (1-q)(-3) & = 3q - 3 \end{cases}$$

Since $3q - 3 > 4q - 5 \; \forall \; q \in [0, 1]$, 1 should defect. However, **there is a better choice if they work together**. But, if both are rational and only self interested, they will end in a worse position. In this situation, we say that D strictly dominates C and C is strictly dominated by D.

Remark 2.1 (Importance)**.** We can see that for both players, their dominant strategy would be (D,D). However, this strategy is *Pareto dominated* by another strategy, (C,C). This is a direct counterexample to the previously Adam Smith's First Welfare Theorem which suggested that competitive markets intrinsically tended to the efficient allocation of resources. However, this is not the case in the Prisoner's dilemma, where the efficient allocation (Pareto dominant) would be (C,C). △

However, we can solve the prisoner's dilemma through a few possible solutions:

- Binding contracts between parties with third party enforcement (e.g. courts)
- Repeated Games
- Adjust the utilities

Different Utiliites

Suppose there is a modified prisoner's dilemma as follows:

		2: C	2: D
1	C	-1, -1	-5, -2
	D	0, -5	-3, -3

Consider the modification to Player 2's defect scenario as a cost of defecting. Note that for Player 1, nothing changes and defecting will still be his best strategy. However, we adjust player 2's perspective as follows:

$$S_1 = D \implies \begin{cases} U_2(C,D) & = -5 \\ U_2(D,D) & = -3 \end{cases} \implies S_2 = D$$

$$S_1 = C \implies \begin{cases} U_2(C,C) & = -1 \\ U_2(D,C) & = -2 \end{cases} \implies S_2 = C$$

Suppose that $P(S_1 = C) = p$. Therefore, we have:

$$\begin{cases} U_2(C,p) = p(-1) + (1-p)(-5) & = 4p - 5 \\ U_2(D,p) = p(-2) + (1-p)(-3) & = p - 3 \end{cases} \Rightarrow 4p - 5 > p - 3 \rightarrow 3p > 2 \implies \begin{cases} p > \frac{2}{3} \implies S_2 = C \\ p < \frac{2}{3} \implies S_2 = D \end{cases}$$

3 Definitions

Definition 3.1 (Strictly Dominates)**.** *A strategy S_i strictly dominates S_i' if $\forall S_j \rightarrow U_i(S_i, S_j) > U_i(S_i', S_j)$. In other words, S_i strictly dominates if it is strictly better no matter what the other player does. We can also say that S_i' is strictly dominated by S_i.*

Definition 3.2 (Strictly Dominant)**.** *If S_i strictly dominates all other S_i', then S_i is strictly dominant. Note that even if there are strategies that strictly dominate another one, that does not mean that there must be a strictly dominant strategy.*

Note that you can attempt to identify strictly dominated strategies and eliminate those options, since they are "bad" ways of playing the game. Instead, you can also attempt to find "good" ways of playing the game.

Definition 3.3 (Expected Value). *Let X be a random variable that takes values X_i with probability p_i s.t. $i = 1..n$. Given this, an expected value would be:*

$$E[X] = p_1 x_1 + ... + p_n x_n$$

Definition 3.4 (Belief of Player). *A belief of a player i is a probability distribution $\sigma_j \in \Delta(S_j)$ that assigns probability $\sigma_j(S_j)$ to strategy profile S_j. Essentially, you are assigning a weight to the chance of you playing a certain strategy. Given the beliefs,* ***expected utlity*** *from S_i given σ_j:*

$$U_i(S_i, \sigma_j) = \sum_{s_j} u_i(s_i, s_j) \cdot \sigma_j(s_j)$$

You are weighting the payoff you get from each possible strategy player 2 may pick by their probability of picking the strategy.

Definition 3.5 (Best Response). *A strategy s_i is a best response to beliefs σ_j if it maximizes $E[U_i]$.*

$$U_i(s_i, \sigma_j) \geq U_i(s_i', \sigma_j) \;\forall\; s_i'$$

$BR_i(\sigma_j)$ is the set of best responses to σ_j.

Example 3.1 (Modified Prisoner's Dillema). Looking again at our modified Prisoner's Dilemma, displayed below:

		2: C	2: D
1	C	-1, -1	-5, -2
	D	0, -5	-3, -3

Let σ_j be s.t. $\sigma_j(C) = p$. Therefore:

$$U_2(C, \sigma_j) = 4p - 5$$

$$U_2(D, \sigma_j) = p - 3$$

We have the following best response:

$$BR_2(\sigma_j) \begin{cases} C & p > \frac{2}{3} \\ D & p < \frac{2}{3} \\ [C, D] & p = \frac{2}{3} \end{cases}$$

□

4 Relationship between Dominance and Best Response

Question 4.1. Is there a relationship between strictly dominated strategies and best responses to a set of beliefs?

Theorem 4.1. *If S_i' if strictly dominated by some S_i, then it can not be a best response to any set of beliefs σ_j.*

Proof Pick any σ_j:

$$U_i(S_i, \sigma_j) = \sum_{s_j} u_i(s_i, s_j) \cdot \sigma_j(s_j)$$

Note that $u_i(s_i, s_j) > u_i(s_i', s_j)$ since s_i strictly dominates s_i' (see Def 3.1). Therefore:

$$U_i(s_i, \sigma_j) > U_i(s_i', \sigma_j)$$

Therefore, s_i' is not the best response to σ_j. ∎

Question 4.2. If s_i' is never a best response, it is strictly dominated

This question does not have a definite answer and can be seen in the following example:

Example 4.1. Suppose the following game:

		B: L	M	R
A	Up	0, 3	0, 1	0, 0
	Down	0, 0	0, 1	0, 3

for σ_A s.t. $\sigma_a(U) = p$.

$$U_B(S_B, \sigma_A) = \begin{cases} 3p & S_B = L \\ 1 & S_B = M \\ 3(1-p) & S_B = R \end{cases}$$

$$BR_B(\sigma_A) = \begin{cases} L & p > \frac{1}{2} \\ R & p < \frac{1}{2} \\ [L, R] & p = \frac{1}{2} \end{cases}$$

We see that M is never considered a best response. However, M is **never** strictly dominated in this game.

Consider a **mixed strategy** by player B chosen by a coin flip, s.t.:

$$\begin{cases} L & \text{if heads} \\ R & \text{if tails} \end{cases}$$

This strategy gives 1.5 no matter what A will end up playing. **This strategy strictly dominates M**. □

Theorem 4.2. *If S_i is **not** a best response, then it is strictly dominated [possibly] by a mixed strategy.*