Application of Mixed Strategies

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1 Expert Diagnosis

Consider a situation where a customer has a problem and has to rely on the diagnosis by an expert, for instance, a car mechanic. A car mechanic inspects a problem with a car and finds that either the car needs a tune-up or it needs a new engine. The customer relies on the advice of the expert but he will realize when the mechanic sold him a tune-up when he really needed a new engine (but not vice-versa).

Therefore, when the mechanic find that the car needs a new engine, he will always recommend a new engine. However, when he finds the car only needs a tuneup, he can either be honest and recommend a tune-up or be dishonest and recommend to replace the engine.

The customer has no reason to second-guess the mechanic when he recommends a tune-up. He may be suspicious when the mechanic recommends tor replace the engine and thus, he can either accept or reject. If he rejects, he needs to get the car fixed at a more honest but more expensive dealer.

1.1 Formalization

We have two players in this game, the Mechanic (M) and Customer (C). The strategies are such that $S_M = \{\text{Honest}, \text{Exagerate}\}$. The customer strategies are such that $S_C = \{\text{Trusting}, \text{Skeptical}\}$. The payoffs are $U(S_C = T) = E$ for engine and 0 for tune-up and $U(S_C = S) = E + X$ for engine and x for tune-up. For the mechanic, the payoffs are π for replacing engine when needed or performing the tune-up when needed. Replacing the engine when a tune-up would have sufficed results in payoff $\pi + y$. r is the probability that the engine does need to be replaced.

Therefore, we have the following payoff matrix:

		C: Trust (q)	C: Skeptical $(1-q)$
M	Honest (p)	$r\pi + (1-r)\pi, -rE - (1-r)0$	$0r + (1-r)\pi, -r(E+x) - (1-r)0$
	Exag $(1-p)$	$r\pi + (1-r)(\pi+y), -rE - (1-r)E$	$0r + (1-r)0, -r(E+x) - (1-r)x$

This payoff matrix simplifies to:

		C: Trust (q)	C: Skeptical $(1-q)$
M	Honest (p)	$\pi, -rE$	$(1-r)\pi, -r(E+x)$
	Exag $(1-p)$	$\pi + (1-r)y, -E$	$0, -rE - x$

Assume that if the customer knows the mechanic is exaggerating, he prefers to go to the dealer.

1.2 Nash Equilibrium

Clearly, there is no pure strategy Nash Equilibrium in this game. Therefore, we must seek for a Mixed Strategy Nash Equilibrium.

We begin by taking the perspective of the mechanic. Assuming that the customer plays a mixed strategy such that:

$$\sigma_C(T) = q$$

We can equate the two utilities to find the point of best response indifference:

$$\begin{aligned} U_M(H, \sigma_C) U_M(E, \sigma_C) \\ q\pi + (1-q)(1-r)(\pi) &= q(\pi + (1-r)y) + (1-q)(0) \\ (1-r)(1-q)\pi &= (1-r)qy \\ \pi &= qy + q\pi \\ q &= \frac{\pi}{\pi + y} \end{aligned}$$

Therefore, we have the mixed strategy of best response indifferent at $\frac{\pi}{\pi+y}$. Note that the probability r does not actually matter in this situation. The mechanic's choice only matters conditional to the car needing an actual tune-up.

Next, taking the perspective of a customer, assuming the mechanic plays a mixed strategy:

$$\sigma_M(H) = p$$

Equating the utilities:

$$\begin{aligned} U_C(T, p) &= U_C(S, p) \\ p(-rE) + (1-p)(-E) &= p(-r(E+x)) + (1-p)(-rE - x) \\ prx &= (1-p)((1-r)E - x) \\ &\dots \\ p &= \frac{(1-r)E - x}{(1-r)(E-x)} \\ &= \frac{E - \frac{x}{1-r}}{E - x} \end{aligned}$$

Therefore, our mixed strategy is $\frac{E - \frac{x}{1-r}}{E-x}$. Our Nash Equilibrium is thus:

$$p = \frac{E - \frac{x}{1-r}}{E - x} \ \wedge \ q = \frac{\pi}{\pi + y}$$

1.3 Comparitors

We can test changing the parameters of the game and watch how the equilibrium changes.

Suppose that $y \downarrow$, in an world where mechanics are regulated more. We note that the only effect is that $q^* \uparrow$. However, this is an interesting result. Despite regulating the mechanics more to make the mechanics more often, this does not actually make the mechanic more honest. Instead, it simply makes the customers more trusting. This is counter to the effect that is initially expected.

2 Reporting a Crime

Suppose that there are many people witnessing a crime. Everyone would report if alone, but does not want to report in a group.

2.1 Formalization

Suppose there are N witnesses, each with $S_i = \{\underline{C}\text{all}, \underline{N}\text{ot Call}\}$. The payoffs are such that:

- v if someone else calls the police, $s_i = N \;\wedge\; s_j \neq N$
- $v - c > 0$ if one calls the police himself, $s_i = C$
- 0 if nobody calls the police, $s = N$

2.2 Nash Equilibrium

We can note that there are N pure strategy Nash Equilibriums, with only one person reporting. This is an asymmetric equilibrium which is almost impossible to coordinate. Instead, we can find a symmetric mixed strategy Nash Equilibrium such that each person reports with a specific probability.

For indifference, our equation is:

$$U(R,p) = U(N,p)$$

$$v - c = v[P(s_j = C)] + 0[P(s_j = N)]$$

Therefore, after solving (using the complement probability), we obtain:

$$\frac{c}{v} = P(s_j = N)$$

Assuming that ever person in N calls with probability p, then $P(s_j = N) = (1-p)^{N-1}$. Therefore, we have that:

$$(1-p)^{N-1} = \frac{c}{v} \implies p = 1 - \sqrt[N-1]{\frac{c}{v}}$$

2.3 Comparitors

Let's suppose that N increases. Note that p itself changes as N, changes. We note that:

$$\begin{aligned} P(s_j = N) &= (1-p)^N \\ &= (\sqrt[N-1]{\frac{c}{v}}^N \\ &= \frac{c}{v}^{\frac{N}{N-1}} \end{aligned}$$

Note that $N \uparrow \to \frac{N}{N-1} \downarrow \implies \frac{c}{v}^{\frac{N}{N-1}} \uparrow$. This is the classic case of the free-rider problem. As well, this can be explained as a pure miscoordination between the players.